Landscape complexity of the elastic manifold

Benjamin McKenna

Harvard University

based on joint work with Gérard Ben Arous and Paul Bourgade (both Courant, NYU)

Workshop on Spin Glasses, Les Diablerets

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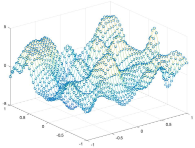
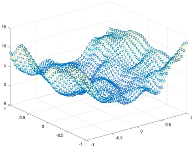
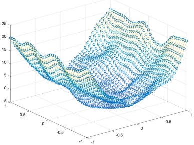
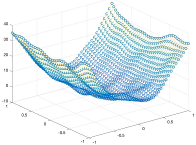





What is this talk about?

- **Landscape complexity:** Let $f_N : \mathbb{R}^N \to \mathbb{R}$ be a sequence of smooth Gaussian random functions ("landscapes"), e.g. Hamiltonians or loss functions. Their (annealed) complexity is

$$\Sigma = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[\mathrm{Crt}(f_N)] := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[\#\{\text{critical points of } f_N\}]$$

i.e.

$$\mathbb{E}[\mathrm{Crt}(f_N)] \approx e^{N\Sigma} \quad \text{for large } N.$$

Also interested in $\Sigma_{\min}$ counting just local minima.

- Today: Σ, $\Sigma_{\min}$ for a special f_N called the "elastic manifold" (not diff. geo.), a classic model with two equivalent interpretations:
  - spin glasses interacting on a lattice ← mostly this one today
  - magnetic interfaces

Why?

$$\mathbb{E}[\mathrm{Crt}_{\min}(f_N)] \approx e^{N\Sigma_{\min}}$$

- Especially if $f_N = f_N(\text{model parameters})$, and therefore $\Sigma_{\min} = \Sigma_{\min}(\text{model parameters})$, good for distinguishing regions in parameter space (at least as a guess), such as
  - (physics, where f_N is a Hamiltonian)

    $$\text{glass phase } (\Sigma_{\min} > 0) \text{ vs. non-glass phase } (\Sigma_{\min} \leq 0),$$

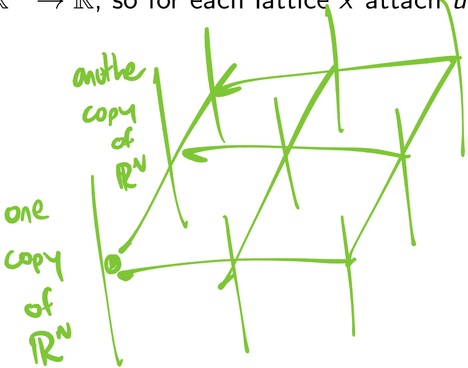
    since loc. min. of Hamiltonian are metastable states, or
  - (data science, where f_N is a loss function)

    $$\text{hard } (\Sigma_{\min} > 0) \text{ vs. easy } (\Sigma_{\min} \leq 0) \text{ to optimize with (S)GD}$$

    since local minima can trap descent algorithms.

Elastic manifold: Spin glasses on a lattice

- Fix once and for all a lattice $\mathcal{L}$, i.e., $\mathcal{L} =$ 
- Spin glasses (continuous, not Ising) are, e.g., random functions $f_{\mathrm{SG}}^{(N)} : \mathbb{R}^N \to \mathbb{R}$, so for each lattice x attach $u(x) \in \mathbb{R}^N$,



Elastic manifold: Spin glasses on a lattice

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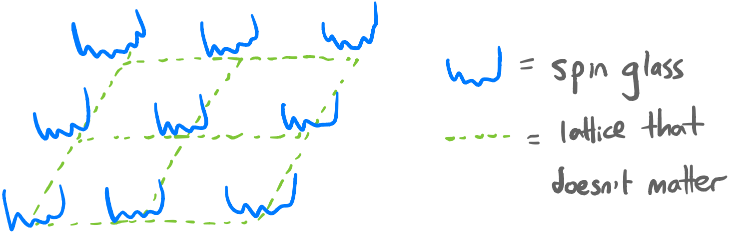
- Spin glasses (continuous, not Ising) are, e.g., random functions $f_{\mathrm{SG}}^{(N)} : \mathbb{R}^N \to \mathbb{R}$, so for each lattice x attach $u(x) \in \mathbb{R}^N$, and consider

$$\mathcal{H}_{\mathrm{prelim}}[\vec{u}] := \sum_{x \in \mathcal{L}} f_{\mathrm{SG}}^{(N)}(u(x)).$$

  Notice $\vec{u} : \mathcal{L} \to \mathbb{R}^N$, i.e., "$\vec{u} \in \mathbb{R}^{N|\mathcal{L}|}$," and $\mathcal{H}_{\mathrm{prelim}} : \{\text{all } \vec{u}\text{'s}\} \to \mathbb{R}$, i.e., "$\mathcal{H}_{\mathrm{prelim}} : \mathbb{R}^{N|\mathcal{L}|} \to \mathbb{R}$," hence in our framework.
- This is a Hamiltonian for $|\mathcal{L}|$ many non-interacting spin glasses:



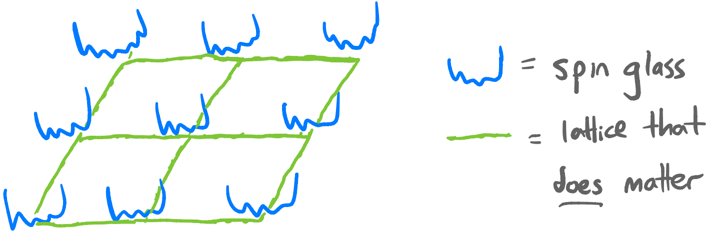
Elastic manifold: Spin glasses on a lattice

- Instead, add interactions as

$$\mathcal{H}_N[\vec{u}] := \sum_{x,y \in \mathcal{L}} t\Delta_{xy} \langle u(x), u(y) \rangle + \sum_{x \in \mathcal{L}} f_{\mathrm{SG}}^{(N)}(u(x))$$

where $t > 0$ is an interaction strength and $\Delta \geq 0$ is your favorite interactions matrix with entries Δ_{xy}.

- This is a Hamiltonian for $|\mathcal{L}|$ many interacting spin glasses:



Elastic manifold: Spin glasses on a lattice

$$\mathcal{H}_N[\vec{u}] = \sum_{x,y \in \mathcal{L}} t\Delta_{xy} \langle u(x), u(y) \rangle + \sum_{x \in \mathcal{L}} f_{\mathrm{SG}}^{(N)}(u(x))$$

- The elastic manifold is a special case of this Hamiltonian, choosing Δ as the (periodic) lattice Laplacian and

$$f_{\mathrm{SG}}^{(N)}(u) = f_{\mathrm{soft}}^{(N)}(u) := \mu \cdot \frac{\|u\|^2}{2} + b \cdot \mathrm{noise}(u)$$

  for "mass" $\mu > 0$ and "disorder" $b > 0$ ("noise" means your favorite "isotropic Gaussian" $\mathbb{R}^N \to \mathbb{R}$, normalized, i.i.d. for different lattice points)

- $f_{\mathrm{soft}}^{(N)}$ is a classic analogue of spherical spin glasses ($(u_i)_{i=1}^N$ are spins with "soft" quadratic constraint $\|u\|^2 = O_{\mathbb{P}}(1)$ instead of "hard" spherical $\|u\|^2 = 1$)
- Fyodorov '04: Complexity of $f_{\mathrm{soft}}^{(N)}$ (special case of our model on a one-point lattice). Auffinger–Ben Arous–Černý '13: Complexity of spherical spin glasses (also one-point).

Elastic manifold: Spin glasses on a lattice

$$\mathcal{H}_N[\vec{u}] = \sum_{x,y\in\mathcal{L}} t\Delta_{xy}\langle u(x), u(y)\rangle + \sum_{x\in\mathcal{L}} \mu \frac{\|u(x)\|^2}{2} + b\cdot \text{noise}(u(x))$$

- Hamiltonian has three competing terms:
  - $t > 0$ "elasticity" (to have low energy, wants to be flat)
  - $\mu > 0$ "mass" (wants to be close to zero)
  - $b > 0$ "disorder" (wants to be rugged)
- These same competing forces act on interfaces between plus and minus regions in a ferromagnet, so this is a classic model for magnetic interfaces.
- Main question: Who wins, as seen in the critical points?
  - If $b = 0$, model is deterministic and quadratic, with only one critical point which is the global minimum $u \equiv 0$.
  - If $t = 0$, model has no interactions, and (via Fyodorov '04) lots of critical points for $\mu < \mu_c(b)$ but few for $\mu \geq \mu_c(b)$.*

Elastic manifold: Main result

$$\mathcal{H}_N[\vec{u}] = \sum_{x,y \in \mathcal{L}} t \Delta_{xy} \langle u(x), u(y) \rangle + \sum_{x \in \mathcal{L}} \mu \frac{\|u(x)\|^2}{2} + b \cdot \text{noise}(u(x))$$

Main theorem, informally (Ben Arous-Bourgade-M. '21)

(fixed lattice, $N \to \infty$, i.e. "Mézard-Parisi scaling") We split (t, μ, b) space into $\{\Sigma > \Sigma_{\min} > 0\}$ (explicit) vs $\{\Sigma = \Sigma_{\min} = 0\}$, the only possibilities, and recognize the μ-marginals of the boundary $0 < \mu_c(t, b) < \infty$ as the physically relevant "Larkin mass(es)."

- Confirms physics results of Fyodorov–Le Doussal '20.
- Proof relies on Kac-Rice formula, which is a standard way to reduce problems about $\mathbb{E}[\text{Crt}]$ into problems about $\mathbb{E}[|\det(W_N)|]$ for W_N some real-symmetric random matrix related to the Hessian.
- We develop general techniques to answer the random matrix question, hence general techniques to study $\mathbb{E}[\text{Crt}]$: the elastic manifold today, but more in the papers, and meant to be broadly applicable.

Elastic manifold: Main result

$$\mathcal{H}_N[\vec{u}] = \sum_{x,y\in\mathcal{L}} t\Delta_{xy} \langle u(x), u(y)\rangle + \sum_{x\in\mathcal{L}} \mu \frac{\|u(x)\|^2}{2} + b \cdot \text{noise}(u(x))$$

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- For this problem, the Kac-Rice random matrix is (basically) $W +$ Laplacian where W is a "Gaussian random band matrix."

- "Larkin mass" comes from the theory of (de)pinning (a certain way magnetic interfaces respond to applied force): Larkin 1970 proposed some simplification of the Hamiltonian which is believed to be good exactly for μ such that $\Sigma = 0$; Larkin's model has no local minima = metastable states ...