Extremal statistics for quadratic forms of GOE/GUE eigenvectors

arXiv: 2208.12206, with László Erdős (ISTA)

Outline:
1. (I) What
2. (II) Who cares
3. (III) Heuristics

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(I) WHAT

Gaussian Orthogonal Ensemble

Recall GOE is a random $N \times N$ real-symmetric matrix whose upper-triangular entries are independent Gaussians. With $(u_i = u_i^{(N)})_{i=1}^N$ GOE eigenvectors, this talk is about large-N distributional limits of

$$\max_{i=1}^{N} \langle u_i, A_N u_i \rangle$$

your favorite sequence $(A_N)_{N=1}^\infty$ of deterministic real-symmetric $N \times N$ matrices

Informal main theorem: Normalize $\|u_i\|_2^2 = 1$, and let $(y_i = y_i^{(N)})_{i=1}^N$ be independent standard Gaussian vectors (so $\|y_i\|_2^2 \approx N$). If

$$\operatorname{rank}(A_N) \leq N^{\frac{1}{2} - \varepsilon}$$

for some $\varepsilon > 0$, then

not deep, just normalization

$$\max_{i=1}^{N} N \langle u_i, A_N u_i \rangle \qquad \text{and} \qquad \max_{i=1}^{N} \langle y_i, A_N y_i \rangle$$

have the same $N \to \infty$ distributional limits.

Benefit: Reduces to Gaussian computations = classical extremal statistics, possible for any A_N. By making some representative choices of A_N, get

Corollary 1 (fixed rank): Fix k, fix $a_1, \dots, a_k$. Let $a = \max_{i=1}^{k} a_i$, $m = \#\{i : a_i = a\}$, and any A_N with $\mathrm{Spec}(A_N) = \{a_1, a_2, \dots, a_k, 0, \dots, 0\}$.

- **Case 1**: If at least one $a_i > 0$, then

$$\frac{N}{2a} \max_{i=1}^{N} \langle u_i, A_N u_i \rangle - \log N + \left(1 - \frac{m}{2}\right) \log\log N + c_m(a_1, \dots, a_k)$$

$$\xrightarrow{N \to \infty} \text{Gumbel in distribution}$$ (i.e. $\mathbb{P}(\text{LHS}_N \le x) \xrightarrow{N\to\infty} \mathbb{P}(\text{RHS} \le x) = e^{-e^{-x}}\ \forall x$)

a named distribution from extremal statistics

with $c_m(a_1, \dots, a_k) = \log\left(\Gamma\left(\frac{m}{2}\right) \sqrt{\prod_{\substack{j=1 \\ a_j \ne a}}^{k} \left(1 - \frac{a_j}{a}\right)}\right)$.

- **Case 2**: If all a_i's negative, then with $\gamma_k = \frac{1}{2}\left(\frac{2}{k\Gamma(k/2)}\right)^{2/k}$,

$$\frac{\gamma_k N^{1+\frac{2}{k}}}{\left(\prod_{j=1}^{k} |a_j|\right)^{\frac{1}{k}}} \max_{i=1}^{N} \langle u_i, A_N u_i \rangle \xrightarrow{N \to \infty} \frac{k}{2}\text{-Weibull in distribution}$$

$$\mathbb{P}(\Psi_{\frac{k}{2}} \le x) = \begin{cases} \exp(-(-x)^{\frac{k}{2}}) & \text{if } x \le 0 \\ 1 & \text{if } x \ge 0 \end{cases}$$

Corollary 2 (diverging rank): Let $A_N = \mathrm{diag}(1, \dots, 1, 0, \dots, 0)$, $\mathrm{rank}(A_N) = N^\alpha$, $0 < \alpha < \frac{1}{2}$.

$$\frac{\sqrt{\log N}}{N^{\alpha/2}} \max_{i=1}^{N} N \langle u_i, A_N u_i \rangle - N^{\frac{\alpha}{2}} \sqrt{\log N} - 2\log N + \frac{\log\log N}{2} + \frac{\log(4\pi)}{2}$$

$$\xrightarrow{N \to \infty} \text{Gumbel in distribution.}$$

Rmks:
- Results also for $\max_{i=1}^{N} |\langle u_i, A_N u_i \rangle|$, and for complex-Hermitian case (GUE). Since GOE evecs are Haar-distributed on the orthogonal group, result equivalent for Haar columns, in particular for evecs of any invariant ensemble (density $\propto e^{-\mathrm{Tr}(V(X))} dX$)
- Rank-one GUE case: Lakshminarayan, Tomsovic, Bohigas, and Majumdar 2008.

II) Who Cares?

1) Delocalization

2) Quantum (unique) ergodicity ← Q(U)E

Idea: This is a sort of distributional, extremal delocalization / QUE.

Rmk: If rank-one $A_N = qq^T$, then $\langle u_i, A_N u_i \rangle = \langle u_i, q \rangle^2$.

1) Mean-field random-matrix evecs are flat $\left(u_i \approx \left(\frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}}\right)\right)$.

Can formalize in various senses, e.g. for any deterministic $\|q = q^{(N)}\| = 1$

$$|\langle u_i, q \rangle| \lesssim \frac{N^{\varepsilon}}{\sqrt{N}} \quad \text{with high probability, for each } i, q \qquad \text{(ST)}$$

or stronger

$$\max_{i=1}^{N} |\langle u_i, q \rangle| \le \frac{N^{\varepsilon}}{\sqrt{N}} \quad \text{with high probability, for each } q \qquad \text{(SE)}$$

These are Size results, either Typical or Extremal.

Complemented by Distributional results like

"$\sqrt{N}\langle u_i, q \rangle$ is asymptotically Gaussian" (DT)

or

"$\max_{i=1}^{N} N\langle u_i, q \rangle^2$ is asymptotically Gumbel" (DE)

These are basically exercises for GOE, from special miracle (later), but universality (proving these for other RMs, like Wigner = real-symmetric with independent non-Gaussian entries) is very non-trivial. History: (incomplete)

	(ST)	(SE)	(DT)	(DE)
Universality for Wigner	Knowles–Yin '13	Benigni–Lopatto '22	Bourgade–Yau '17	open!

Open: For $(u_i)_{i=1}^N$ evecs of a (even any!) non-Gaussian Wigner matrix,

$$\frac{N}{2}\max_{i=1}^{N}\langle u_i, q\rangle^2 - \log N + \frac{1}{2}\log\log N + \log\sqrt{\pi} \xrightarrow{N\to\infty} \text{Gumbel in distribution.}$$

Even just for $q = e_1$, so $\langle u_i, q\rangle^2 = u_i(1)^2$. Even subleading orders – state of the art is $\frac{N}{2}\max_{i=1}^{N}\langle u_i, q\rangle^2 = O(\log N)$, Benigni–Lopatto '22.

② Many disordered quantum systems have

$$\lim_{i,j\to\infty}\langle \psi_i, A\psi_j\rangle = \delta_{ij} f(A)$$

where
- (ψ_i) are eigenfunctions of some Hamiltonian
- A is any deterministic operator (observable) in some good class
- f is a model-dependent specific functional on this class

either for most pairs i,j (QE) or, stronger, for all pairs i,j (QUE).

Examples:
- Laplacian on some Riemannian manifolds (influential QUE conjecture of Rudnick–Sarnak '94, still open)
- Regular graphs, both deterministic and random.
- Random matrices (our setting)

Sort of like higher-rank delocalization. Can also ask in senses (ST), (SE), (DT), (DE). History in random matrices:

(ST): Bourgade–Yau '17

(SE): Bourgade–Yau–Yin '20; Benigni '21; Cipolloni–Erdős–Schröder '22

(DT): Benigni–Lopatto '22; Cipolloni–Erdős–Schröder '22

} universality

(DE): this talk } Gaussian computation

universality open (harder than rank-one case?)

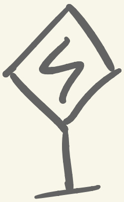
III HEURISTICS (why true? why up to $N^{1/2}$?)

Miracle: GOE eigenvectors are exactly distributed as independent standard Gaussian vectors after Gram-Schmidt.

Idea: Maybe Gram-Schmidt is basically just a rescaling here, since

(a) high-dimensional Gaussian vectors are almost orthogonal

(b) norm of a Gaussian vector concentrates around $\sqrt{N}$.



Idea is correct for a few vectors jointly, but Gram-Schmidt error can add up after a lot of them. See works of Tiefeng Jiang around 2005 on exactly how far this can be pushed for various observables/metrics.

Hence our rank restriction. Note some restriction on A_N is necessary: If $A_N = Id$, then $\max_{i=1}^{N} N\langle u_i, A_N u_i\rangle = N$ is deterministic, but $\max_{i=1}^{N} \langle y_i, A_N y_i\rangle$ fluctuates, so conclusion fails.

Important notation for rest of talk:

Sizes: $y_{ij} \sim 1$

$\sqrt{N}\,\gamma_{ij} \sim 1$

- $K = K_N = \operatorname{rank}(A_N)$.
    - Rotation-invariance $\Rightarrow$ WLOG $A_N = \operatorname{diag}(a_1, \dots, a_K, 0, \dots, 0)$.

      For simplicity here, set $a_i \equiv 1$.
- Let $Y \in \mathbb{R}^{N \times N}$ have IID standard Gaussian rows y_i.
- Let $\Gamma \in \mathbb{R}^{N \times N}$ have rows $\gamma_i = GS(y_i) \approx \frac{y_i}{\sqrt{N}}$.

  i.e. Γ is Haar-orthogonal and $(\gamma_1, \dots, \gamma_N) \stackrel{d}{=} (u_1, \dots, u_N)$.
- For γ_{ij}, y_{ij}, first index is GS index (more important).

Index-swap miracle: (Rank-one for clarity) If $A_N = e_1 e_1^T$,

$$\max_{i=1}^{N} \langle u_i, A_N u_i \rangle \overset{d}{=} \max_{i=1}^{N} \langle \gamma_i, A_N \gamma_i \rangle = \max_{i=1}^{N} \gamma_{i1}^2 \overset{d}{=} \max_{i=1}^{N} \gamma_{1i}^2$$

since Haar measure is the same if you flip rows and columns.
That is, instead of first K entries of all N Gram-Schmidt vectors, only have to handle all N entries of first K Gram-Schmidt vectors!

Rmk: ① Huge benefit since GS error compounds for later ones.

② Completely fails for Wigner matrices.

The good coupling: Let

$$P_N := \max_{j=1}^{N} \sum_{i=1}^{K} y_{ij}^2 \overset{d}{=} \max_{i=1}^{N} \langle y_i, A_N y_i \rangle$$

$$Q_N := \max_{j=1}^{N} \sum_{i=1}^{K} (\sqrt{N}\gamma_{ij})^2 \overset{d}{=} \max_{i=1}^{N} N \langle u_i, A_N u_i \rangle$$

Main theorem (formal): If $K_N = \operatorname{rank}(A_N) \leq N^{1/2-\varepsilon}$, then

$$P_N - Q_N \xrightarrow{N\to\infty} 0 \text{ in probability.}$$

Proof sketch (why $N^{1/2}$?): Deterministically

$$\left| \max_{j=1}^{N} |A_j + B_j| - \max_{j=1}^{N} |A_j| \right| \leq \max_{j=1}^{N} |B_j|.$$

Thus

$$|P_N - Q_N| = \left| \max_{j=1}^{N} \sum_{i=1}^{K} (y_{ij} - y_{ij} + \sqrt{N}\gamma_{ij})^2 - \max_{j=1}^{N} \sum_{i=1}^{K} y_{ij}^2 \right|$$

$$\leq \max_{j=1}^{N} \left| \sum_{i=1}^{K} \left(2 y_{ij} (\sqrt{N}\gamma_{ij} - y_{ij}) + (\sqrt{N}\gamma_{ij} - y_{ij})^2 \right) \right|$$

Jiang 2005: $\max_{j=1}^{N} \max_{i=1}^{K} |\sqrt{N}\gamma_{ij} - y_{ij}| \lesssim \sqrt{\frac{K}{N}}$ (probably tight up to log factors)

Here: $\max_{j=1}^{N}$ only adds log factors, so ignore it.

Naive strategy: Triangle inequality. y_{ij} order-one, so get

$$|P_N - Q_N| \lesssim K\left(2\sqrt{\tfrac{K}{N}} + \left(\sqrt{\tfrac{K}{N}}\right)^2\right) \sim K\sqrt{\tfrac{K}{N}}$$

which tends to zero if $K \ll N^{1/3}$.

Better strategy: For second term, $K\left(\sqrt{\frac{K}{N}}\right)^2$ already allows $K \ll N^{1/2}$ (nothing to cancel anyway). For first term, CLT instead of triangle inequality morally saves $\frac{1}{\sqrt{K}}$, so get size $K \cdot \sqrt{\frac{K}{N}} \cdot \frac{1}{\sqrt{K}}$, which also leads to $K \ll N^{1/2}$.

Technical: Actually implement via high-moment expansion ($\sim \frac{1}{\varepsilon}$ moments for $K \sim N^{\frac{1}{2}-\varepsilon}$). The $\sqrt{N}\gamma_{ij} - y_{ij}$ is annoying, since has projection of y_i onto γ_j for $j<i$, which has norms in denominator. Carefully replace with projection of y_i onto y_j, which is better for first high-moment expansion, with a second high-moment expansion. All this is graphically bookkept, like Feynman diagrams. Bulk of paper.

IV SUMMARY

1. Quadratic forms of GOE/GUE eigenvectors behave as if the eigenvectors were independent Gaussian vectors, under a rank restriction.
2. This allows for explicit computation and limit laws.
3. Like a distributional, extremal form of delocalization/QUE.
4. Proof: Gauss-Gram-Schmidt, index-swap miracle, graphically bookkept high-moment expansions.
5. Restriction $\mathrm{rank}(A_N) \ll N^{1/2}$ is the limit for CLT techniques?
6. Open: Universality (even for rank-one, even for second order).